EQUATIONS OF A FLUID BOUNDARY LAYER WITH COUPLE STRESSES

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Equations of a plane boundary layer of a viscous incompressible fluid with couple stresses, asymmetric stress tensor and with the inner inertia of the particles taken into account, are considered. Numerous variants of the plane boundary layer equations are investigated and their invariant group theoretic properties obtained. Boundary layer equations are discussed in connection with two problems, one concerned with the flow around a flat plate and the other with a totally submerged stream. General problems of the theory of fluids with couple stresses were investigated in [1 and 2].

1. General system of equations of motion of a viscous, incompressible fluid with couple stresses, has the form

$$\nabla\cdot\mathbf{v}=0,\quad \frac{d\mathbf{v}}{dt}=-\frac{1}{\rho}\nabla p+2\nu\nabla\cdot(\nabla\mathbf{v})^d+\nu_r\nabla\times[2\boldsymbol{\omega}-\nabla\times\mathbf{v}]$$
$$I\frac{d\boldsymbol{\omega}}{dt}=2\nu_r(\nabla\times\mathbf{v}-2\boldsymbol{\omega})+c_0\nabla(\nabla\cdot\boldsymbol{\omega})+2c_d\nabla\cdot(\nabla\boldsymbol{\omega})^d+2c_a\nabla\cdot(\nabla\boldsymbol{\omega})^a \tag{1.1}$$

Here ρ denotes the bulk density, p is the pressure, I is a scalar constant of dimension equal to that of the moment of inertia of unit mass, $\mathbf{v}$ is the velocity vector of a point, $\boldsymbol{\omega}$ is the vector describing the mean angular velocity of rotation of the particles of which a point of the continuum is composed, ν is the kinematic Newtonian viscosity, ν_r is the kinematic rotational viscosity, c_0, c_d and c_a are the coefficients of the couple stress viscosity, $d(\ldots)/dt$ denotes the total differential with respect to time, ∇ is the three-dimensional grad, $(\nabla\mathbf{v})^d$ and $(\nabla\omega)^d$ are the symmetric parts of the corresponding dyads, finally, $(\nabla\mathbf{v})^a$ and $(\nabla\boldsymbol{\omega})^a$ are the antisymmetric dyads.

For the plane case Eqs. (1.1) become, in the dimensionless form,

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=-\frac{\partial p}{\partial x}+\left(\frac{1}{R}+\frac{1}{R_r}\right)\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)+\frac{2}{R_r}\frac{\partial\omega_z}{\partial y}$$
$$\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=-\frac{\partial p}{\partial y}+\left(\frac{1}{R}+\frac{1}{R_r}\right)\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right)-\frac{2}{R_r}\frac{\partial\omega_z}{\partial x} \tag{1.2}$$
$$\frac{\partial\omega_z}{\partial t}+v_x\frac{\partial\omega_z}{\partial x}+v_y\frac{\partial\omega_z}{\partial y}=-\frac{4E}{R_r}\omega_z+\frac{2E}{R_r}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)+\frac{E}{R_c}\left(\frac{\partial^2\omega_z}{\partial x^2}+\frac{\partial^2\omega_z}{\partial y^2}\right)$$
$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$$

$$R=\frac{Vl}{\nu},\quad E=\frac{V^2}{I},\quad R_r=\frac{Vl}{\nu_r},\quad R_c=\frac{Vl^3}{\gamma},\quad \gamma=\frac{c_a+c_d}{I} \tag{1.3}$$

where V and l denote the characteristic velocity and length, respectively.

In the curvilinear [3] orthogonal coordinates q_1 and q_2 with the Lamé coefficients h_1 and h_2, Eqs. (1.2) become

$$\frac{\partial v_1}{\partial t}+\frac{v_1}{h_1}\frac{\partial v_1}{\partial q_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial q_2}+\frac{v_2}{h_1h_2}\left(v_1\frac{\partial h_1}{\partial q_2}-v_2\frac{\partial h_2}{\partial q_1}\right)= \tag{1.4}$$

$$=-h_1\frac{\partial p}{\partial q_1}+\left(\frac{1}{R}+\frac{1}{R_r}\right)\left[\frac{1}{h_1^2}\frac{\partial^2 v_1}{\partial q_1^2}+\frac{1}{h_2^2}\frac{\partial^2 v_1}{\partial q_2^2}+\frac{1}{h_1h_2}\frac{\partial(h_2/h_1)}{\partial q_1}\frac{\partial v_1}{\partial q_1}+\right.$$

$$+\frac{1}{h_1h_2}\frac{\partial(h_1/h_2)}{\partial q_2}\frac{\partial v_1}{\partial q_2}+\frac{2}{h_1^2h_2}\frac{\partial h_1}{\partial q_2}\frac{\partial v_2}{\partial q_1}-\frac{2}{h_1h_2^2}\frac{\partial h_2}{\partial q_1}\frac{\partial v_2}{\partial q_2}+$$

$$+\frac{1}{h_1}\frac{\partial}{\partial q_1}\left(\frac{1}{h_1h_2}\frac{\partial h_2}{\partial q_1}\right)v_1+\frac{1}{h_2}\frac{\partial}{\partial q_2}\left(\frac{1}{h_1h_2}\frac{\partial h_1}{\partial q_2}\right)v_1+\frac{1}{h_1}\frac{\partial}{\partial q_1}\left(\frac{1}{h_1h_2}\frac{\partial h_1}{\partial q_2}\right)v_2-$$

$$\left.-\frac{1}{h_2}\frac{\partial}{\partial q_2}\left(\frac{1}{h_1h_2}\frac{\partial h_2}{\partial q_1}\right)v_2\right]+\frac{2}{R_r}\frac{1}{h_2}\frac{\partial\omega_3}{\partial q_2}$$

$$\frac{\partial v_2}{\partial t}+\frac{v_1}{h_1}\frac{\partial v_2}{\partial q_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial q_2}-\frac{v_1}{h_1h_2}\left(v_1\frac{\partial h_1}{\partial q_2}-v_2\frac{\partial h_2}{\partial q_1}\right)=-\frac{1}{h_2}\frac{\partial p}{\partial q_2}+$$

$$+\left(\frac{1}{R}+\frac{1}{R_r}\right)\left[\frac{1}{h_1^2}\frac{\partial^2 v_2}{\partial q_1^2}+\frac{1}{h_2^2}\frac{\partial^2 v_2}{\partial q_2^2}+\frac{1}{h_1h_2}\frac{\partial(h_2/h_1)}{\partial q_1}\frac{\partial v_2}{\partial q_1}+\frac{1}{h_1h_2}\frac{\partial(h_1/h_2)}{\partial q_2}\frac{\partial v_2}{\partial q_2}-\right.$$

$$-\frac{2}{h_1^2h_2}\frac{\partial h_1}{\partial q_2}\frac{\partial v_1}{\partial q_1}+\frac{2}{h_1h_2^2}\frac{\partial h_2}{\partial q_1}\frac{\partial v_1}{\partial q_2}+\frac{1}{h_1}\frac{\partial}{\partial q_1}\left(\frac{1}{h_1h_2}\frac{\partial h_2}{\partial q_1}\right)v_2+\frac{1}{h_2}\frac{\partial}{\partial q_2}\left(\frac{1}{h_1h_2}\frac{\partial h_1}{\partial q_2}\right)v_2-$$

$$\left.-\frac{1}{h_1}\frac{\partial}{\partial q_1}\left(\frac{1}{h_1h_2}\frac{\partial h_1}{\partial q_2}\right)v_1+\frac{1}{h_2}\frac{\partial}{\partial q_2}\left(\frac{1}{h_1h_2}\frac{\partial h_2}{\partial q_1}\right)v_1\right]-\frac{2}{R_r}\frac{1}{h_1}\frac{\partial\omega_3}{\partial q_2}$$

$$h_2\frac{\partial v_1}{\partial q_1}+h_1\frac{\partial v_2}{\partial q_2}+v_1\frac{\partial h_2}{\partial q_1}+v_2\frac{\partial h_1}{\partial q_2}=0$$

$$\frac{\partial\omega_3}{\partial t}+\frac{v_1}{h_1}\frac{\partial\omega_3}{\partial q_1}+\frac{v_2}{h_2}\frac{\partial\omega_3}{\partial q_2}+\frac{\omega_3}{2h_1}\frac{\partial v_1}{\partial q_1}+\frac{\omega_3}{2h_2}\frac{\partial v_2}{\partial q_2}+\frac{v_1\omega_3}{2h_1h_2}\frac{\partial h_2}{\partial q_1}+\frac{v_2\omega_3}{2h_1h_2}\frac{\partial h_1}{\partial q_2}=$$

$$=-\frac{4E}{R_r}\omega_3+\frac{2E}{R_r}\frac{1}{h_1h_2}\left[\frac{\partial(h_2v_2)}{\partial q_1}-\frac{\partial(h_1v_1)}{\partial q_2}\right]+\frac{E}{R_c}\left[\frac{1}{h_1^2}\frac{\partial^2\omega_3}{\partial q_1^2}+\frac{1}{h_2^2}\frac{\partial^2\omega_3}{\partial q_2^2}+\right.$$

$$\left.+\frac{1}{h_1h_2}\frac{\partial(h_2/h_1)}{\partial q_1}\frac{\partial\omega_3}{\partial q_1}+\frac{1}{h_1h_2}\frac{\partial(h_1/h_2)}{\partial q_2}\frac{\partial\omega_3}{\partial q_2}\right]$$

Following [3] we put

$$q_1=x,\quad q_2=\frac{y}{\sqrt{R}},\quad v_1=v_x,\quad v_2=\frac{v_y}{\sqrt{R}},\ \omega_3=\omega_z\sqrt{R},\quad h_1=1+\frac{y}{r\sqrt{R}},\quad h_2=1 \tag{1.5}$$

Let us now insert (1.5) into (1.4) and put $R\to\infty$. As the result, we obtain four possible types of the boundary layer equations depending on the relations between R, R_r, R_c and E.

1. If R, R_r, E and R_c/R are of the same order, we obtain

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+(\nu+\nu_r)\frac{\partial^2 u}{\partial y^2}+2\nu_r\frac{\partial\omega}{\partial y}$$

$$\frac{\partial p}{\partial y}=0,\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\quad u=v_x,\quad v=v_y,\ \omega=\omega_z \tag{1.6}$$

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=-\frac{4\nu_r}{I}\omega-\frac{2\nu_r}{I}\frac{\partial u}{\partial y}+\frac{\gamma}{I}\frac{\partial^2\omega}{\partial y^2}$$

in the dimensional variables. Boundary layer equations contain, in this case, the terms characterizing the asymmetry of the stress dyad, the couple stresses and the inertia of the rotating particles.

2) If R, R_r and E are of the same order, and $ER \ll R_c$, we have the system (1.6) in which the last equation is replaced by

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = -\frac{4\nu_r}{I}\omega - \frac{2\nu_r}{I}\frac{\partial u}{\partial y} \tag{1.7}$$

This case corresponds to the absence of the couple stresses in the fluid.

3) If R, R_r and $\sqrt{R_c}$ are of the same order provided that $E \gg R$, we obtain (1.6), the last equation of which has the form

$$0 = -4\nu_r\omega - 2\nu_r\frac{\partial u}{\partial y} + \gamma\frac{\partial^2\omega}{\partial y^2} \tag{1.8}$$

This corresponds to the case when the inertia of the rotating particles can be neglected.

4) If R, R_r and R_c / E are of the same order, and $E \ll R$, then the last equation of (1.6) is replaced by

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{\gamma}{I}\frac{\partial^2\omega}{\partial y^2} \tag{1.9}$$

which corresponds to the case when the couple stress is dominant during the rotation of the particles.

2. Following [4 – 6] we shall investigate the invariant group theoretic properties of the equations of the boundary layer of the fluid with couple stresses, when the motion is steady.

A) Let us consider the first type of the boundary layer equations. System (1.6) in its normal form is given by

$$\alpha_y = \frac{1}{\nu+\nu_r}\left(uu_x + v\alpha + \frac{1}{\rho}p_x - 2\nu_r\beta\right), \quad u_y = \alpha, \quad \omega_y = \beta$$
$$p_y = 0, \quad v_y = -u_x, \quad \beta_y = \frac{I}{\gamma}\left(u\omega_x + v\beta + \frac{4\nu_r}{I}\omega + \frac{2\nu_r}{I}\alpha\right) \tag{S_1}$$

Here and in the following the subscripts x and y following the quantities u, v, p, ω, α and β denote the differentiation with respect to these variables.

Quantities x, y, u, v, ω, p, α and β are regarded as coordinates defining a point in the space E_8. Let us obtain a group G of transformations of E_8 with the following infinitesimal operator

$$X = \xi_x\frac{\partial}{\partial x} + \xi_y\frac{\partial}{\partial y} + \xi_u\frac{\partial}{\partial u} + \xi_v\frac{\partial}{\partial v} + \xi_\alpha\frac{\partial}{\partial \alpha} + \xi_\omega\frac{\partial}{\partial \omega} + \xi_\beta\frac{\partial}{\partial \beta} + \xi_p\frac{\partial}{\partial p}$$

where ξ_x, ξ_y, ... are functions of $x, y, p, u, v, \alpha, \beta$ and ω, respectively. Let E_{20} be the continuation of E_8 with respect to all the derivatives in u, v, ω, p, α and β and let X^+ be the continuation of X.

System (S_1) will admit the group G, if and only if the conditions [4 – 6]

$$X^+\left[\alpha_y - \frac{1}{\nu+\nu_r}\left(uu_x + v\alpha + \frac{1}{\rho}p_x - 2\nu_r\beta\right)\right] = 0$$
$$X^+(u_y - \alpha) = 0, \quad X^+(\omega_y - \beta) = 0, \quad X^+p_y = 0, \quad X^+(v_y + u_x) = 0 \tag{2.1}$$
$$X^+[\beta_y - (Iu\omega_x + Iv\beta + 4\nu_r\omega + 2\nu_r\alpha)\gamma^{-1}] = 0.$$

hold. The system of equations (2.1) on the manifold (S_1) decomposes, yielding a system of defining equations the general solution of which has the form

$$\xi_x = ax + b_1, \quad \xi_p = 2ap + b_2, \quad \xi_u = au, \quad \xi_w = a\omega \tag{2.2}$$
$$\xi_\alpha = a\alpha, \quad \xi_\beta = a\beta, \quad \xi_y = b_3\varphi(x), \quad \xi_v = b_3u\varphi'(x), \quad \varphi'(x) = d\varphi / dx$$

where $\varphi(x)$ is an arbitrary function of x.

Thus the system (S_1) admits the infinite group G. Since ξ_x, ξ_y, ξ_u, ξ_v, ξ_w and ξ_p are independent of α and β, instead of considering the space E_8 we can consider only the space E_6, in which the Lie algebra of the group G is generated by the following abbreviated basis operators

$$X_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2p\frac{\partial}{\partial p} + \omega\frac{\partial}{\partial \omega}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial p} \tag{2.3}$$

together with the set of operators of the form

$$X_4 = \varphi(x)\frac{\partial}{\partial y} + u\varphi'(y)\frac{\partial}{\partial v} \tag{2.4}$$

It can be shown that the subalgebra of the operators (2.4) will be the ideal of the complete Lie algebra of the admissible system (S_1).

To find the substantially different invariant solutions of (S_1), we shall construct an optimal system of operators (2.3) generating the algebra of residues on the ideal of (2.4).

Computations [4 and 5] yield the following one-parameter subgroups.

Subgroup H_1 with the operator X_1. This subgroup has the associated set of the independent invariants

$$J_1 = u/x, \quad J_2 = v, \quad J_3 = p/x^2, \quad J_4 = \omega/x, \quad J_5 = \xi = y$$

Inserting the values of u, v, p and ω obtained from the above invariants into (S_1), we obtain another system of equations which we shall call (S_1 / H_1)

$$\rho(\nu+\nu_r)J_1'' = \rho J_1^2 + \rho J_2 J_1' + 2J_3 - 2\rho\nu_r J_4', \quad J_3' = 0$$

$$J_1 + J_2' = 0, \quad \gamma J_4'' = IJ_1J_4 + IJ_2J_4' + 4\nu_r J_4 + 2\nu_r J_1'$$

Here and in the following the prime will denote differentiation with respect to ξ.

Subgroup H_2 with the operator X_2. The complete set of invariants has the form

$$J_1 = u, \quad J_2 = v, \; J_3 = p, \; J_4 = \omega, \; J_5 = \xi = y$$

The corresponding system (S_1 / H_2) can be written in the form $(\nu+\nu_r)J_1'' = J_2J_1' - 2\nu_r J_4'$

$$J_3' = 0, \quad J_2' = 0 \quad \gamma J_4'' = IJ_2J_4' + 4\nu_r J_4 + 2\nu_r J_1'$$

Subgroup H_3 with the operator X_3. The complete set of invariants consists of

$$J_1 = u, \quad J_2 = v, \quad J_3 = \omega, \; J_4 = x, \quad J_5 = \xi = y$$

Invariant solution cannot [4 and 5] be constructed on this subgroup.

Subgroup H_4 with the operator $X = \partial(...)/\partial x + \theta(...)/\partial p$. Here we have the following independent invariants

$$J_1 = u, \quad J_2 = v, \quad J_3 = p - x, \; J_4 = \omega, \quad J_5 = \xi = y$$

System (S_1 / H_4) has the form

$$\rho(\nu+\nu_r)J_1'' = \rho J_2J_1' - 2\rho\nu_r J_4', \quad J_3' = 0$$

$$J_2' = 0, \quad \gamma J_4'' = IJ_2J_4' + 4\nu_r J_4 + 2\nu_r J_1'$$

Subgroup H_5 with the operator X_4. Complete set of the independent invariants is given by $J_1 = u, \quad J_2 = v - uy\varphi'/\varphi, \quad J_3 = p, \quad J_4 = \omega, \quad J_5 = \xi = x$

System (S_1/H_5) is completely integrable and yields the following solution

$$u = \frac{c_2}{\varphi(x)}, \quad v = \Phi(x) + \frac{c_2 y\varphi'(x)}{\varphi(x)}, \quad p = c_1 - \frac{c_2^2\rho}{\varphi^2}, \quad \omega = C_3\exp\left[-\frac{4\nu_r}{c_2 I}\int\varphi(x)\,dx\right]$$

where $\Phi(x)$ is arbitrary, c_1, c_2 and c_3 are constants of integration.

It can be shown that the boundary layer equations (1.7) and (1.8) of the second and third type admit the same group G as Eq. (S_1). Therefore their solutions can be obtained from the systems (S_1 / H_1) — (S_1 / H_5), in which γ and I are assumed equal to zero.

Table 1.

Subgroup	Operator	Subgroup	Operator
H_1	X_3	H_8	X_2
H_2	X_4	H_9	$X_2 + X_3$
H_3	X_5	H_{10}	$X_1 + mX_2$
H_4	$X_3 + X_4$	H_{11}	$2X_1 + X_2 + X_4$
H_5	$X_3 + X_5$	H_{12}	$3X_1 + X_2 + X_5$
H_6	$X_4 + X_5$	H_{13}	X_6
H_7	$X_3 + X_4 + X_5$		

B) Let us now consider the fourth type of the boundary layer equations. The system (1.9) in its normal form is

$$\alpha_y = \frac{1}{\nu+\nu_r}\left(uu_x + v\alpha + \frac{1}{\rho}p_x - 2\nu_r\beta\right), \quad u_y = \alpha, \ \omega_y = \beta \qquad (S_2)$$

$$p_y = 0, \quad v_y = -u_x, \quad \beta_y = I\gamma^{-1}(u\omega_x + v\beta)$$

In this case a general solution of the defining equations of the type (2.1) is obtained in the form

$$\xi_x = ax + b_1, \quad \xi_y = (1-a)y + b_2\varphi(x), \quad \xi = (3a-2)u$$

$$\xi_v = (a-1)y + b_2 u\varphi'(x), \quad \xi_p = (6a-4)p + b_3, \quad \xi_\omega = (4a-3)\,\omega + b_4.$$

The optimal system of the operators of the Lie algebra corresponding to the fourth type of the boundary layer equations is given in Table 1.

We see that for the subgroups H_1, H_2, H_4 and H_{13} the invariants are the same as those obtained for the first type of the boundary layer. Subgroups H_3 and H_6 have no invariant solutions and the remaining subgroups shall be discussed below.

Subgroup H_5. The complete set of independent invariants consists of

$$J_1 = u, \quad J_2 = v, \quad J_3 = p, \quad J_4 = \omega - x, \quad J_5 = \xi = y$$

System of equations (S_2 / H_5) has the form

$$(\nu+\nu_r)J_1'' = J_2J_1' - 2\nu_rJ_4', \quad J_2' = 0, \quad J_3' = 0, \quad \gamma J_4'' = IJ_1 + IJ_2J_4'$$

Subgroup H_7. The corresponding complete set of the independent invariants is

$$J_1 = u, \quad J_2 = v, \quad J_3 = p - x, \quad J_4 = \omega - x, \quad J_5 = \xi = y$$

System (S_2 / H_7) becomes

$$\rho(\nu+\nu_r)J_1'' = \rho J_2J_1' + 1 - 2\rho\nu_rJ_4', \quad J_2' = 0$$

$$J_3' = 0, \quad \gamma J_4'' = IJ_1 + IJ_2J_4'$$

Subgroup H_8. The complete set of the independent invariants has the form

$$J_1 = uy, \quad J_2 = vy, \quad J_3 = py^4, \quad J_4 = \omega y^3, \quad J_5 = \xi = x$$

and the corresponding system of equations (S_2 / H_8) is

$$\rho J_1J_1' - 2\rho J_2J_1 = 6\rho(\nu+\nu_r)J_1 - 3\rho\nu_rJ_4, \quad J_2 = J_1'$$

$$J_3 = 0, \quad \gamma J_4 = IJ_1J_4' + 3IJ_2J_4$$

Subgroup H_9. Here we have the following independent invariants:

$$J_1 = u\exp(2x), \quad J_2 = v\exp(x), \quad J_3 = p\exp(4x)$$

$$J_4 = \omega\exp(3x), \quad J_5 = \xi = y\exp(-x)$$

System (S_2 / H_9) has the form

$$\rho(\nu+\nu_r)J_1'' = -2\rho J_1^2 - \rho\xi J_1J_1' + \rho J_2J_1' - 4J_3, \quad J_3' = 0$$

$$-\xi J_1' - 2J_1 + J_2 = 0, \quad \gamma J_4'' = IJ_2J_4' - 3IJ_1J_4 - I\xi J_1J_4'$$

Putting $J_1 = \varphi'$ and $J_4 = \psi$, we obtain

$$(\nu+\nu_r)\varphi''' + 2\varphi'^2 - \varphi\varphi'' + 4c_1 = -2\nu_r\psi', \quad p = c_1\exp(-4x)$$

$$\gamma\psi'' = I(2\varphi' - \xi\varphi' + \xi\varphi'')\psi' - 3I\varphi'\psi$$

Subgroup H_{10}. The complete set of the independent invariants is

$$J_1 = ux^{-1+2m}, \quad J_2 = vx^m, \quad J_3 = px^{-2(1-2m)}, \quad J_4 = \omega^{-1+3m}, \quad J_5 = \xi = yx^{-m}$$

The system (S_2 / H_{10}) becomes

$$(\nu+\nu_r)\varphi''' + (1-m)\varphi\varphi'' - (1-2m)\varphi'^2 - 2(1-2m)c_1 = -2\nu_r\psi'$$

$$\gamma\psi'' = I(1-3m)\varphi'\psi - I(1-m)\varphi\psi', \quad p = c_1\rho x^{2(1-2m)} \qquad (2.5)$$

Subgroup H_{11}. This subgroup has the following independent invariants:

$$J_1 = u, \quad J_2 = vx^{1/3}, \quad J_3 = p - \ln x^{1/3}, \; J_4 = \omega x^{1/3}, \quad J_5 = \xi = yx^{-1/3}$$

and the system (S_2 / H_{11}) has the form

$$2\rho(\nu + \nu_r)\varphi''' + \rho\varphi\varphi'' - 1 = -4\rho\nu_r\psi'$$

$$2\gamma\psi'' = -2I\xi\varphi'\psi' - I\varphi\psi', \quad p = c_1 + \ln x^{1/3}$$

Subgroup H_{12}. The complete set of the independent invariants is

$$J_1 = ux^{-1/3}, \quad J_2 = vx^{1/3}, \quad J_3 = px^{-2/3}, \; J_4 = \omega - \ln x^{1/3}, \quad J_5 = \xi = yx^{-1/3}$$

The system (S_2 / H_{12}) will have the form

$$3\rho(\nu + \nu_r)\varphi''' - \rho\varphi'^2 + 2\rho\varphi'\varphi'' + 2c_1 = -6\rho\nu_r\psi'$$

$$\gamma\psi'' = I\varphi' + I(\xi - 2)\varphi'\psi', \quad p = c_1 x^{2/3}$$

3. We shall now consider the problem of the flow of a viscous, incompressible fluid with couple stresses, around a plane semi-infinite plate. We shall assume that the following boundary conditions hold: $u = v = 0$ when $y = 0$ (3.1)

$$\lim u(x, y) = U(x) = cx^n, \quad \lim \omega(x, y) = 0 \quad \text{as } y \to \infty$$

At the rigid wall, ω may assume one of the following values:

$$\omega = 0 \quad \text{when } y = 0, \qquad \omega_y = -u_{yy} \quad \text{when } y = 0 \tag{3.2}$$

Pressure p is obtained from [3]

$$p_x = -\rho UU' = -nc^2\rho x^{2n-1} \tag{3.3}$$

Conditions (3.1) – (3.3) hold only for the self-similar solution of (2.5) in which the following substitution should be made:

$$m = (1 - n) / 2, \quad c_1 = -c^2 / 2$$

From (2.5) we obtain

$$2(\nu + \nu_r)\varphi''' + (n + 1)\varphi\varphi'' + 2n(c^2 - \varphi'^2) = -4\nu_r\psi'$$

$$2\gamma\psi'' + I(n + 1)\varphi\psi' - I(3n - 1)\varphi'\psi = 0 \tag{3.4}$$

Boundary conditions (3.1) will become

$$\varphi(0) = \varphi'(0) = 0, \quad \varphi'(\infty) = c, \quad \psi(\infty) = 0 \tag{3.5}$$

Relations (3.2) can be written in the form

$$\psi(0) = 0 \quad \text{or} \quad 2\psi(0) + \varphi''(0) = 0 \tag{3.6}$$

If the fluid has constant velocity when $y = \infty$, then Eqs. (3.4) become

$$2(\nu + \nu_r)\varphi''' + \varphi\varphi'' = -4\nu_r\psi', \quad 2\gamma\psi'' + I(\varphi\psi)' = 0 \tag{3.7}$$

From (3.4) with (3.5) and the first condition of (3.6) taken into account, we see that $\omega \equiv 0$, i.e. when $\omega \not\equiv 0$ a solution of (3.4) is possible, provided that the conditions (3.5) and the second condition of (3.6) hold.

4. Let us consider the problem of a totally submerged flow of fluid with couple stresses, using Eqs. (2.5). In accordance with the condition of the conservation of impulse of the stream

$$\rho \int_{-\infty}^{\infty} u^2\, dy = M = \text{const}$$

we must put $m = 2/3$ and $c_1 = 0$.

Relations (2.5) yield equations describing the submerged flow

$$3(\nu + \nu_r)\varphi''' + (\varphi\varphi')' = -6\nu_r\psi', \; 3\gamma\psi'' + I\varphi\psi' + 3I\varphi'\psi = 0, \quad \rho \int_{-\infty}^{\infty} \varphi'^2\, d\xi = M \tag{4.1}$$

Boundary conditions for φ will become [3]

$$\varphi(0) = \varphi''(0) = \varphi'(\infty) = 0 \tag{4.2}$$

By symmetry we have, for ψ,

$$\psi''(0) = 0 \tag{4.3}$$

and at infinity, we adopt one of the following conditions:

$$\psi(\infty) = 0, \quad 2\psi(\infty) + \varphi''(\infty) = 0 \tag{4.4}$$

We note that Eqs. (4.1) with the condition (4.2) and the first condition of (4.4), coincide with the equations of motion for a submerged stream of a Newtonian viscous fluid ($\psi = 0$). If, on the other hand, the conditions (4.2) and (4.3) together with the second condition of (4.4) are taken, then the solution of the problem on the submerged stream with couple stresses leads to the process of integrating (4.1).

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EQUILIBRIUM FIGURES OF A ROTATING LIQUID CYLINDER

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The equilibrium figures of a homogeneous right cylinder kept together by surface tension forces are considered. As we know, the only equilibrium cylindrical figure in the absence of rotation is a right circular cylinder (this shape corresponds to minimal surface energy). Such a cylinder remains an equilibrium figure with rotation about the axis of symmetry of the normal cross section. However, as will be shown below, new equilibrium figures in the form of right cylinders with nth order ($n = 2, 3, \ldots$) axes of symmetry arise for certain